Derivation Of
Trigonometric Sight Reduction and Sailings
An 11th-Grade View

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I. INTRODUCTION.

Electronic tools having advanced so far in reliability and ease, the JN course cannot be justified today on practical grounds. The naval academy was right to discard celestial navigation as a required course. Even when done well, it yields only approximate results. As one letter-writer to the New York Times said it last year, the most useful function of a sextant and a slide rule aboard a lifeboat today is as paddles.

Power squadron members study celestial navigation for a different reason. We appreciate the history of inquiry and the beauty of the celestial sphere. We sail among the stars to sense connections with our environment and our past.

But do we study well? Chapter 9 of the JN manual describes the NASR method for solving the navigational triangle. Page 9-1 assures members that the NASR table, based on spherical trigonometry, does the "higher" math for you in computing a celestial line of position. Page 9-5 counsels not to worry about mastering the mathematics underlying the tables. Again, the sailings taught in chapters 15-17 are for impractical rhumb-line courses.

This is unsatisfying and obstructive. It does not do to stifle the natural curiosity of sailors, especially where just high school mathematic notions are involved.

The following assumes only that you know

* 4 of the 6 basic trigonometric functions of plane geometry, the sine, cosine, tangent, and cotangent,

* the sine of the complement of an angle is the cosine of the angle and vice-versa, and

* the tangent of the complement of an angle is the cotangent of the angle and vice-versa,

which are reviewed for convenience in the last section. Assuming as navigators do that the earth is a sphere, this paper will show the geometric relations of the Mercator chart, the navigational triangle, and great circle courses. We will also prove geometric methods of sight reduction and great circle sailings accessible to a general audience.

In the process will be shown the solution to any spherical triangle given any 3 of the six parts (3 angles, 3 sides), except if given only 3 angles. The latter can be solved using polar triangles, a slightly more nuanced topic which will not be shown here.

One problem I have not yet run to ground yet, as will be seen, is the construction of the auxiliary table to the NASR reduction method on pages 316-17 of the Almanac. That is only a footnote though, because the methods developed here obviate the need for that table.

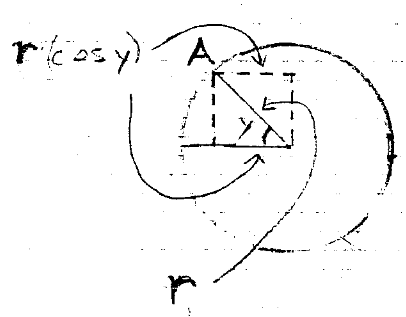
Aesthetically I prefer trig tables and a slide rule for computation. These were the tools of Bowditch and Napier. Like the practical midshipmen though, most people use easy and accurate electronic calculators. The choice is the reader's.

II. MERCATOR CHARTS

Chapter 13 of the JN manual explains the concept of Mercator charts, without explaining the simple formula they are based on. The distance d between two meridians along a parallel at latitude L is given by:

$$d = e \bullet \cos L$$

where e = the distance between the two meridians at the equator. Why? Consider the following side-view figure of the earth with radius r sliced vertically along a meridian:



The point A is at latitude L, so $L = \angle y$. The length of the equator -- the circumference -- is $2\pi r$. Because its radius is $(r \bullet \cos L)$, the length of the circular parallel that goes through A all the way around the earth is $2\pi(r \bullet \cos L)$. $\therefore$ the ratio of the length of this parallel to the equator is

$$\begin{aligned} &2\pi r / 2\pi(r \bullet \cos L) \\ = &1/(\cos L). \end{aligned}$$

Now, as explained with figures 13-3 to 13-6 of the JN manual, the Mercator chart is constructed by expanding both the latitude and longitude by the same factor. The above demonstrates, that this factor is just 1/(cos L). Otherwise stated, multiply the equatorial distance by cos L and you have the distance along the parallel.

For example, at L = 60°N, cos L = ½. ∴ on the globe, the distance between two meridians along the 60°N parallel is half the distance between the same two meridians along the equator.

III. THE NAVIGATIONAL TRIANGLE

A great circle is defined as the intersection of a sphere and a plane that passes through the center. Any two different planes through the center ∴ intersect in a diameter; their corresponding great circles accordingly intersect at the endpoints 180° apart. That is, any two great circles bisect each other.

A spherical triangle is a triangle on a sphere formed by great circles. Thus, meridians are great circles, and parallels of latitude generally are not.

Consider the two figures reproduced below. The left one is the navigational triangle of figure 9-4 on page 9-4 of the JN manual. The right one is a spherical triangle with right angle C, taken from Seymour & Smith, "Solid Geometry" (MacMillan Co, New York, 1943), page 191.

The left figure shows the navigational spherical triangle divided into two right spherical triangles marked as "I" and "II."

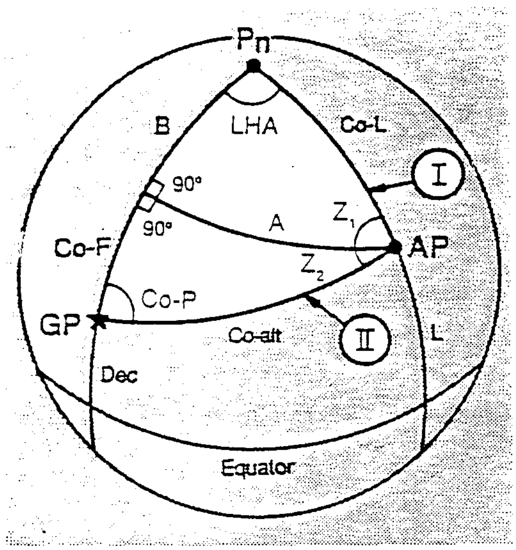
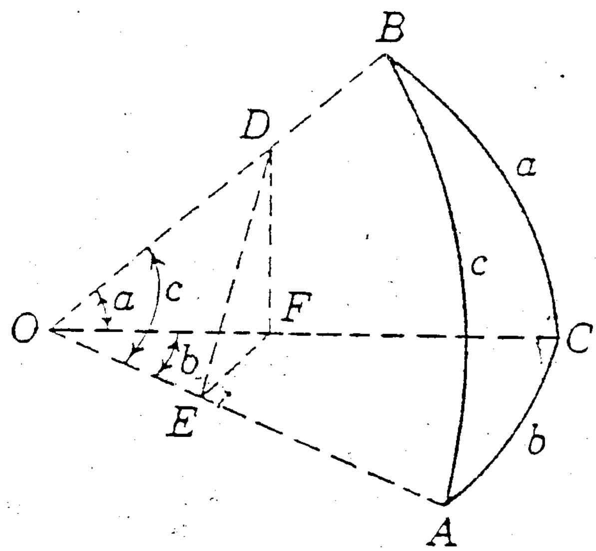


Figure 9-4



The right figure is cleaner and easier to work with. So, re-letter the sides and angles of triangle I on the left to match those of the right figure.

$$\begin{aligned} 90^\circ &\mapsto \angle C \\ \angle \mathrm{LHA} &\mapsto \angle A \\ \text{Co-L} &\mapsto c \\ A &\mapsto a \\ B &\mapsto b \\ \angle Z_1 &\mapsto \angle B. \end{aligned}$$

The problem of determining A, B, and $\angle Z_1$ in the left figure reduces to solving for a, b, and $\angle B$, given $\angle A$ and c in the right figure. We do this using Napier's rules ## 1, 2, and 6, developed below.

Similarly, the sides and angles of triangle II can be re-lettered:

$$\begin{aligned} 90^\circ &\mapsto \angle C \\ A &\mapsto b \\ \text{Co-F} = \text{Co-}(B + \text{dec}) &\mapsto a \\ \angle Z_2 &\mapsto \angle A \\ \angle \text{Co-P} &\mapsto \angle B \\ \text{Co-alt} &\mapsto c. \end{aligned}$$

The problem of determining Co-alt and $\angle Z_2$ in the left figure reduces to solving for c and $\angle A$ in the right figure, given a and b. We do this using Napier's rules ## 5 and 2, developed below.

IV. NAPIER'S RULES.

Napier discovered 10 lovely rules for solving right spherical triangles with right angle C and sides < 180°:

1:	$\sin a = \sin c \bullet \sin \angle A$	6:	$\sin b = \sin c \bullet \sin \angle B$
2:	$\tan b = \tan c \bullet \cos \angle A$	7:	$\tan a = \tan c \bullet \cos \angle B$
3:	$\tan a = \sin b \bullet \tan \angle A$	8:	$\tan b = \sin a \bullet \tan \angle B$
4:	$\cos \angle A = \sin \angle B \bullet \cos a$	9:	$\cos \angle B = \sin \angle A \bullet \cos b$
5:	$\cos c = \cos a \bullet \cos b$	10:	$\cos c = \cot \angle A \bullet \cot \angle B.$

These are nice because a lot of the spherical triangles encountered in navigation are right. The triangle of a great circle course, the meridian of a point on the course, and the equator is right, as are triangles I and II of the NASR sight reduction method. We also use rule # 5 to prove the law of cosines.

We will prove four of Napier's rules here, ## 1, 2, 5, and 6. Consider first the case where all sides < 90°. Construct plane triangle DEF in the right figure above as explained below. We will then show that DEF is a right triangle with ∠F = 90°, ∠DEF = ∠A, and all the faces of tetrahedron ODEF being right triangles.

Let O be the center of the earth with radii OA, OB, and OC. Through D, any point on OB, pass plane DEF at right angles to OA. DE and EF are both perpendicular to OA by definition. By convention, a spherical angle is measured by any plane angle of its dihedral angle. In other words, by definition, ∠DEF is the measure of dihedral ∠B-OA-C which in turn measures ∠A, and ∠DEF = ∠A.

Plane COA is perpendicular to both planes BOC and DEF, since ∠C = 90° and DEF was constructed to be so. So DF is perpendicular to plane COA, as well as to segments OF and FE.

∴ all 4 face triangles of tetrahedron ODEF are right triangles. But the 3 face angles radiating from O respectively = sides a, b, and c of spherical triangle ABC. That means the various ratios of the sides of this tetrahedron yield the trigonometric functions of the sides of ABC. E.g. sin a = DF/OD, tan b = EF/OE, etc.

Napier's rules ## 1, 2, 6, and 5 are now derived. Suppose OD = 1. Then

$$\begin{aligned} \sin a &= DF \\ &= DE \cdot \sin \angle DEF \\ &= \sin c \cdot \sin \angle A. \end{aligned}$$

This is rule 1. For rule 2:

$$\begin{aligned} \tan b &= EF/OE \\ &= (EF/ED) \div (OE/ED) \\ &= (\cos \angle A) \div (\cot c) \\ &= \tan c \cdot \cos \angle A. \end{aligned}$$

Rule 6 follows by the same process as rule 1 except instead of constructing plane triangle DEF through OB perpendicular to OA with one of its angles = ∠A, construct a plane triangle through OA perpendicular to OB with one of its angles = ∠B. Result:

$$\sin b = \sin c \cdot \sin \angle B.$$

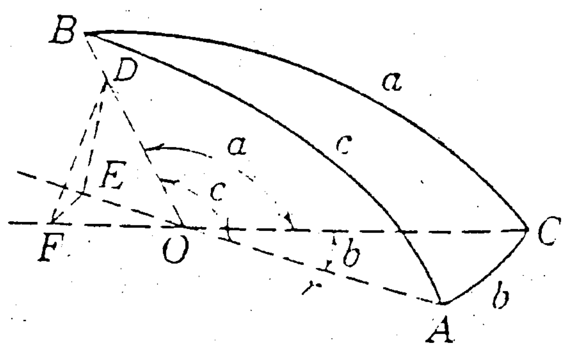
For rule 5:

$$\begin{aligned} \cos c &= OE \\ &= OF \cdot \cos b \\ &= \cos a \cdot \cos b. \end{aligned}$$

It can be seen by inspection that sides A, Co-L, and Co-H in the navigational triangle all < 90°. ∴ their cosines are all positive. Applying rule # 5 to right triangles I and II in the navigational triangle ⇒ the cosines of B and Co-F must also be positive, so they too < 90°. Confirming this, the NASR tables only allow values < 90° of all the sides of triangles I and II.

So for reducing sights Napier's rules need not be proved for triangles with sides exceeding 90°. But the law of cosines, proved below, is used by navigators for other purposes such as determining long great circle distances. So for completeness we will prove the rules for these triangles with sides up to 180°.

The following diagram, taken from page 192 of Seymour & Smith, is used to prove the rules where a > 90°, c > 90°, and b < 90°. The diagram is a little confusing, but I can't draw and it is the only model I have. The point F should be shown a little to the right of where it is shown because both ∠DFO and ∠DFE turn out to be right angles.



Anyway, the proofs above for rules ## 1, 2, 6, and 5 all work using this diagram, once it is noted that

$$\angle\ DEF = 180^\circ - \angle A$$
$$\angle\ DOF = 180^\circ - c$$
$$\angle\ DOF = 180^\circ - a$$
$$\angle\ EOF = b$$

and the following identities are recalled for all x:

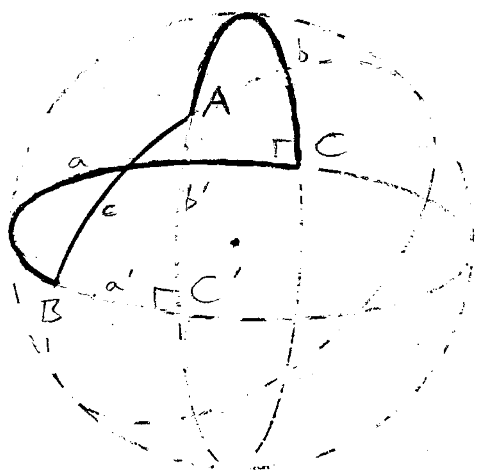
$$\sin x = \sin (180^\circ - x)$$
$$\cos x = - \cos (180^\circ - x)$$
$$\tan x = - \tan (180^\circ - x).$$

The next diagram, taken from Shapiro, "Mathematics Encyclopedia" (Doubleday & Co, Garden City, 1977), page 188, is useful to show the rules in triangle ABC, where ∠C = 90°, a > 90°, b > 90°, and c < 90°. Since intersecting great circles bisect each other, note that:

$a = 180° - a'$ $\angle BAC = 180° - \angle BAC'$

$b = 180° - b'$ $\angle ABC = 180° - \angle ABC'$

All 10 of Napier's rules will have been proved as to triangle ABC′ by the end of this paper. Substituting via the identities for a′ and b′ in the 10 formulas for triangle ABC′ yields the same 10 formulas for triangle ABC.



Inspection of this diagram will also satisfy the reader there are no right spherical triangles with all 3 sides a > 90°.

As for a triangle with 2 sides = 90°, inspection will show the sides opposite them also = 90°. With the exception of the formulas involving tan 90°, which are nonsense, Napier's rules become trivial. The same is true for the triangle with all 3 sides = 90°.

V. CONSTRUCTION OF THE NASR TABLES.

These rules can reduce a sight. Thus in triangle I of the left figure on page 3:

$$\sin A = \sin(\text{Co-L}) \cdot \sin \angle LHA \qquad \text{(rule \# 1)}$$
$$= \cos L \cdot \sin \angle LHA.$$

$$\tan B = \tan(\text{Co-L}) \cdot \cos \angle LHA \qquad \text{(rule \# 2)}$$
$$= \cot L \cdot \cos \angle LHA.$$

$$\sin B = \sin(\text{Co-L}) \cdot \sin \angle Z1 \qquad \text{(rule \# 6)}$$
$$\sin \angle Z_1 = \sin B \div \sin(\text{Co-L})$$
$$= \sin B \div \cos L.$$

In triangle II:

$$\cos(\text{Co-}H_C) = \cos(\text{Co-F}) \cdot \cos A \qquad \text{(rule \# 5)}$$
$$\sin H_C = \sin F \cdot \cos A$$
$$= \sin(B + \text{dec}) \cdot \cos A.$$

$$\tan A = \tan(\text{Co-}H_C) \cdot \cos \angle Z_2 \qquad \text{(rule \# 2)}$$
$$\cos \angle Z_2 = \tan A \div \cot H_C.$$

In triangles I and II:

$$\angle Z = \angle Z_1 + \angle Z_2.$$

Inspection of the NASR tables suggests they were constructed using these formulas.

The tables yield results only for whole degrees of arguments. Corrections are needed for a refined value of H_C. The auxiliary table is used.

I am not clear on the derivation of the auxiliary table except (referring to the left figure on page 3) to note that the responses

$$\text{corr } 1 = \Delta F \cdot \sin P$$
$$= \Delta(\text{Co-F}) \cdot \cos(\text{Co-P})$$

and

$$\text{corr } 2 = \Delta A \cdot \cos Z_2,$$

where ΔF, $\Delta(\text{Co-F})$, and ΔA are the differences between F, Co-F, and A, and their respective nearest whole degrees. That is, each correction is the cosine of an oblique angle in triangle II multiplied by an increment of its adjacent side.

Perhaps the corrections result from applications of Napier's rule # 2 ($\tan b = \tan c \cdot \cos \angle A$) on triangles formed by dropping perpendiculars from points on sides Co-F and A to side Co-alt = H_C.

Or they may consist of approximations derived from plane right triangles, the first with oblique angle Co-P and adjacent side Co-F, and the second with oblique angle Z_2 and adjacent side A.

I am not sure.

Page 284 of the almanac explains that in any event the auxiliary table is capable of introducing error in H_c of up to $2'$.

The trigonometric method by contrast is limited only by the accuracy of your slide rule.

VI. THE LAW OF COSINES

The law of cosines is an even nicer tool, applicable to all spherical triangles, whether right or not. It holds:

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \angle A,$$

for any three sides a, b, and c, and any $\angle A$ opposite side a.

Recall that the sine of an angle is the cosine of its complement, and vice-versa, to see that the formulas on page 9-11 are both applications of the law of cosines to the navigational triangle. As will be seen below the law is also useful to determine the great circle distance between two points.

First we prove it, and a different but very similar proposition, for right spherical triangles. Then we generalize it to all spherical triangles.

Consider our friend, the right-hand figure on page 3 above:

$$\begin{aligned} & \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos \angle A \\ = {} & OE/OF \cdot OE + EF/OF \cdot DE \cdot EF/DE \\ = {} & (OE)^2/OF + (EF)^2/OF \\ = {} & ((OE)^2 + (EF)^2)/OF \\ = {} & (OF)^2/OF \\ = {} & OF \\ = {} & \cos a. \end{aligned}$$

This is the law of cosines for oblique $\angle A$. To prove it for oblique $\angle B$, draw the triangle we constructed to prove Napier rule # 6, and go through the same process. To prove it for right $\angle C$, since $\cos 90^\circ = 0$, note

$$\begin{aligned} &\cos b \cdot \cos a + \sin b \cdot \sin a \cdot \cos \angle C \\ &= \cos b \cdot \cos a \\ &= \cos c, \end{aligned}$$

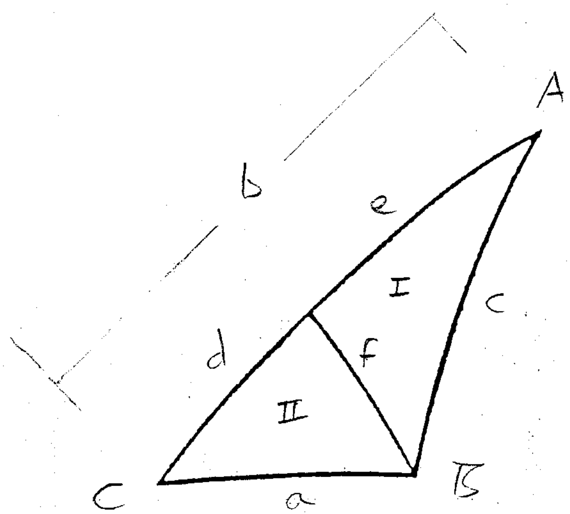
by Napier rule # 5.

Similar reasoning leads to a slightly different result, used below to prove the law of cosines generally:

$$\begin{aligned} &\sin b \cdot \cos c - \cos b \cdot \sin c \cdot \cos \angle A \\ &= EF/OF \cdot OE - OE/OF \cdot DE \cdot EF/DE \\ &= EF \cdot OE/OF + EF \cdot OE/OF \\ &= 0. \end{aligned}$$

Call this the "zero lemma," for want of something more imaginative.

Now, consider the arbitrary oblique spherical triangle shown in a figure taken from Seymour & Smith at page 208:



First, drop a perpendicular f from $\angle B$ to side b, as shown forming two right triangles, I and II. Then,

$$\begin{aligned}
&\cos b \bullet \cos c + \sin b \bullet \sin c \bullet \cos \angle A \\
&= \cos (d + e) \bullet \cos c + \sin (d + e) \bullet \sin c \bullet \cos \angle A.
\end{aligned}$$

Using the formulas for the sine and cosine of the sum of 2 angles developed at the end of this paper, this expression:

$$\begin{aligned}
= &(\cos d \bullet \cos e - \sin d \bullet \sin e) \bullet \cos c \\
&+ (\sin d \bullet \cos e + \cos d \bullet \sin e) \bullet \sin c \bullet \cos \angle A
\end{aligned}$$

$$\begin{aligned}
= &(\cos d \bullet \cos e \bullet \cos c) \\
&- (\sin d \bullet \sin e \bullet \cos c) \\
&+ (\sin d \bullet \cos e \bullet \sin c \bullet \cos \angle A) \\
&+ (\cos d \bullet \sin e \bullet \sin c \bullet \cos \angle A)
\end{aligned}$$

$$\begin{aligned}
= &\cos d\ ((\cos e \bullet \cos c) + (\sin e \bullet \sin c \bullet \cos \angle A)) \\
&- \sin d\ ((\sin e \bullet \cos c) - (\cos e \bullet \sin c \bullet \cos \angle A)).
\end{aligned}$$

Apply the above-proved law of cosines and the zero lemma to right triangle I to obtain:

$$\begin{aligned}
&= (\cos d \bullet \cos f) - (\sin d \bullet 0) \\
&= \cos d \bullet \cos f \\
&= \cos a,
\end{aligned}$$

the last step following from Napier rule # 5. This is the law of cosines.

VII. AN EXAMPLE.

This example is taken from Problem 2(b) on page 9-17 of the JN manual. A sun sight is given with

$$\begin{aligned}
\text{assumed } L &= 49^\circ\text{N} \\
\angle \text{LHA} &= 65^\circ\ 18.1' \\
&\approx 65^\circ \\
\text{dec} &= 13^\circ\ 46.3'\text{N}.
\end{aligned}$$

Using the power squadron's NASR form and tables, Appendix F computes $H_C = 26^\circ\ 40'$, $Z = 100.1^\circ$, and $Z_N = 260^\circ$. Do trigonometric methods yield the same result? First, using Napier's rules, the slide rule yields:

$$
\begin{aligned}
\sin A &= \sin(\text{Co-L}) \bullet \sin \text{LHA} && \text{(rule \# 1)}\\
&= \cos L \bullet \sin \text{LHA}\\
&= \cos 49^\circ \bullet \sin 65^\circ\ 18.1'\\
&= (.656)(.91)\\
&= .597\\
A &= 36.6^\circ
\end{aligned}
$$

$$
\begin{aligned}
\tan B &= \tan(\text{Co-L}) \bullet \cos \text{LHA} && \text{(rule \# 2)}\\
&= \cot L \bullet \cos \text{LHA}\\
&= \cot 49^\circ \bullet \cos 65^\circ\ 18.1'\\
&= (.871)(.418)\\
&= .364\\
B &= 20.0^\circ
\end{aligned}
$$

$$
\begin{aligned}
\sin \angle Z_1 &= \sin B \div \sin(\text{Co-L}) && \text{(rule \# 6)}\\
&= \sin B \div \cos L\\
&= \sin 20.0^\circ \div \cos 49^\circ\\
&= (.342)/(.656)\\
&= .522\\
\angle Z_1 &= 31.5^\circ
\end{aligned}
$$

$$
\begin{aligned}
\sin H_C &= \cos(\text{Co-H}_C) && \text{(rule \# 5)}\\
&= \cos(\text{Co-F}) \bullet \cos A\\
&= \sin F \bullet \cos A\\
&= \sin(B + 13.8^\circ) \bullet \cos A\\
&= \sin(20.0^\circ + 13.8^\circ) \bullet \cos 36.6^\circ\\
&= \sin 33.8^\circ \bullet \cos 36.6^\circ\\
&= (.557)(.803)\\
&= .446\\
H_C &= 26.5^\circ\\
&= 26^\circ\ 30'
\end{aligned}
$$

$$
\begin{aligned}
\cos \angle Z_2 &= \tan A \div \tan(\text{Co-H}_C) && \text{(rule \# 2)}\\
&= \tan A \div \cot H_C\\
&= \tan 36.6^\circ \div \cot 26.5^\circ\\
&= (.744)/(2.0)\\
&= .372\\
\angle Z_2 &= 68.3^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle Z &= \angle Z_1 + \angle Z_2\\
&= 31.5^\circ + 68.3^\circ\\
&= 99.8^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle Z_N &= 260.2^\circ\\
&= 260^\circ\ 12'.
\end{aligned}
$$

Computing the same result with the law of cosines gives for H_C:

$$
\begin{aligned}
\sin H_C &= \sin L \bullet \sin dec + \cos L \bullet \cos dec \bullet \cos LHA \\
&= \sin 49^\circ \bullet \sin 13^\circ\ 46.3' \\
&\quad + \cos 49^\circ \bullet \cos 13^\circ\ 46.3' \bullet \cos 65^\circ\ 18.1' \\
&= (.755)(.238) + (.656)(.971)(.418) \\
&= .1795 + .2665 \\
&= .446 \\
H_C &= 26^\circ\ 29',
\end{aligned}
$$

and for Z:

$$
\begin{aligned}
\sin dec &= \sin L \bullet \sin H_C + \cos L \bullet \cos H_C \bullet \cos Z \\
\sin 13^\circ\ 46.3' &= \sin 49^\circ \bullet \sin 26^\circ\ 29' \\
&\quad + \cos 49^\circ \bullet \cos 26^\circ\ 29' \bullet \cos Z \\
.238 &= (.755)(.446) + (.656)(.895) \bullet \cos Z \\
&= .337 + .588 \bullet \cos Z \\
-.099 &= .588 \bullet \cos Z \\
\cos Z &= -(.099)/(.588) \\
&= -.1685 \\
Z &= 99^\circ\ 42' \\
Z_N &= 260^\circ\ 18'.
\end{aligned}
$$

The results by these methods are quite close to those found on the NASR form in Appendix F, and I would suggest more accurate. Between Napier and the law of cosines, the latter is faster.

-an method

VIII. GREAT CIRCLE SAILINGS

Great circle courses are the shortest. Using trig rules, great circle sailings are simple. This is so even though great circle headings constantly change. With a known initial position and course you can compute any of the following if you know the other two: (1) any other position, (2) the course to it or from it, and (3) the distance to it. We will start with (2).

A. At any two points the sines of the headings are inversely proportional to the cosines of the latitudes.

The following trick tells the navigator where to steer at any point on the route, knowing only the course at some other point and the present latitude. It follows from Napier's rule # 4, which was stated above and will now be proved.

In the right figure on page 3

$$\begin{aligned} \cos \angle A &= EF/ED \\ &= (OF \bullet \sin b)/\sin c. \\ &= OF \bullet (\sin c)\bullet(\sin \angle B)/(\sin c) \qquad \text{(rule \# 6)} \\ &= OF \bullet \sin \angle B \\ &= \cos a \bullet \sin \angle B, \end{aligned}$$

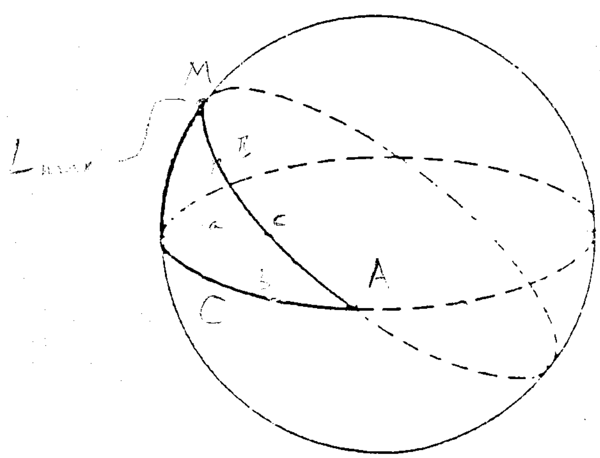
which is rule # 4.

So, suppose first you start from A on the equator on a course that passes through B. Segment AC = b is on the equator. What should be your heading at point B?

Your latitude L at B is segment BC = a.

Note also that your heading at point B is the vertical angle of ∠B, so it = ∠B.

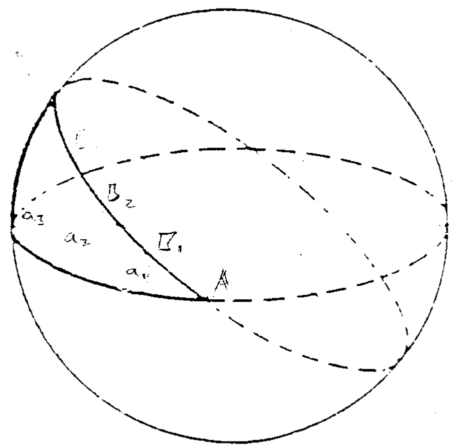
Extend your course past B to a point, say M, where the course is nearest the pole, and the latitude reaches its maximum and begins to diminish. Call this latitude L_{max}. Draw a meridian from M. The resultant spherical triangle, consisting of segment AM, the equator, and the meridian, is a spherical triangle with right angles at each endpoint of the meridian (M and on the equator), and with one angle = ∠A.



By rule 4, in that triangle, $\cos \angle A = \cos L_{max} \bullet \sin 90° = \cos L_{max}$, so $\angle A = L_{max}$. So, still assuming our course began on the equator, the problem reduces to solving for ∠B if we know a and ∠A.

But also by rule 4, in triangle ABC, $\cos \angle A = \cos a \bullet \sin \angle B = \cos L_{max}$, so $\sin \angle B = (\cos L_{max}) \div (\cos L)$, and $\angle Q = \angle B$, which was to be proved.

What if our course began off the equator?



Starting at A on the equator, pick any series of points B_1, B_2, B_3,... on the course. For each of them

$$\begin{aligned}\cos L_{max} &= \cos a_1 \bullet \sin \angle Q_1\\ &= \cos a_2 \bullet \sin \angle Q_2\\ &= \cos a_3 \bullet \sin \angle Q_3\\ &= \ldots,\end{aligned}$$

which was to be proved.

Thus for example, consider the course from St. Augustine to the Bull Light in chapter 14 of the JN manual. By inspection of either the gnomonic or Mercator charts, reproduced below, its maximum latitude $L_{max} = 51° 45'$ so $\sin L_{max} = .619$. Also by inspection the latitudes of the course at various longitudes are:

	L_O	L
St. Augustine	81° 19′	29° 54′
	80°	31° 00′
	70°	38° 10′
	60°	42° 25′
	50°	47° 15′
	40°	49° 40′
	30°	51° 10′
	20°	51° 42′
Bell Light	10° 18′	51° 35′

Applying the above formula the true headings Q at each of these latitudes should be:

Gnomonic Projection: Great Circle Plots as Straight Line

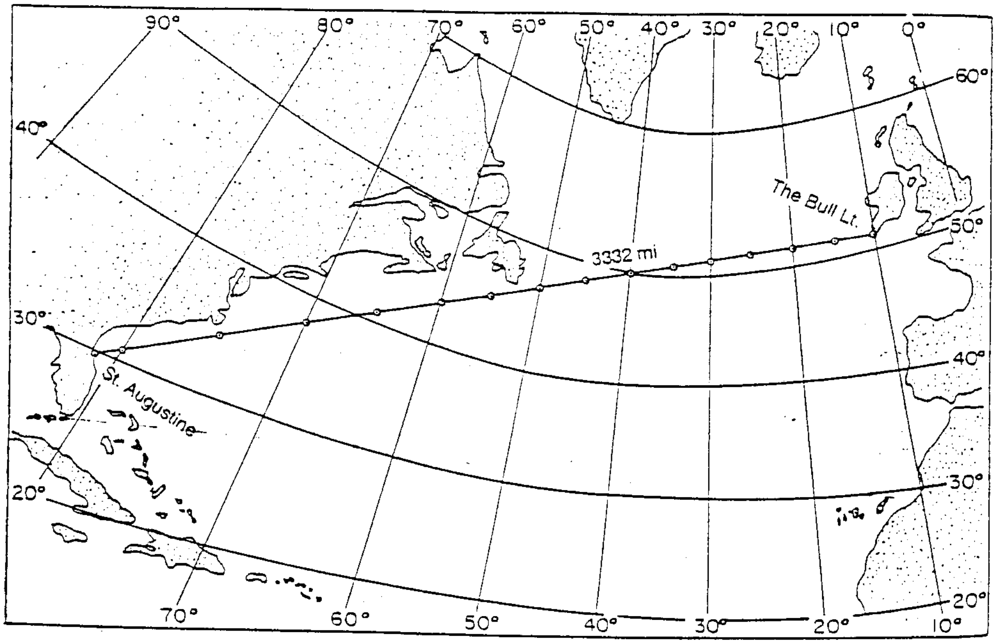


Figure 14-2

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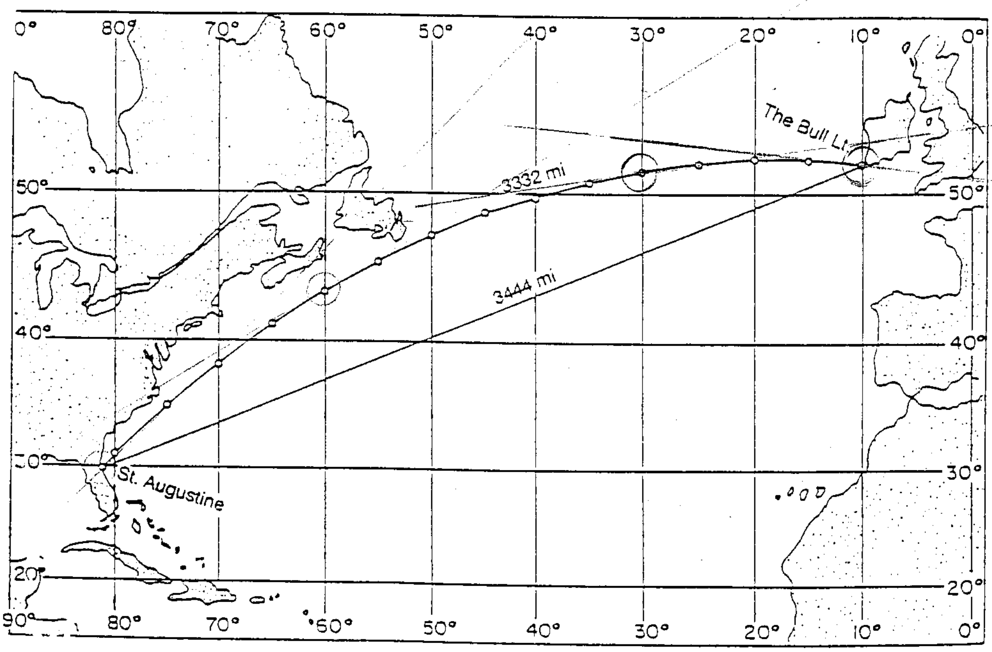
Mercator Chart: Great-circle Course vs. Rhumb-line Course



C 091
C 096

Figure 14-3

L	Q
29° 54′	$\sin \angle Q = (\cos 51° 45') \div (\cos 29° 54') = (.619)/(.874) \Rightarrow \angle Q = 45°$
31° 00′	$\sin \angle Q = (\cos 51° 45') \div (\cos 31° 00') = (.619)/(.857) \Rightarrow \angle Q = 46°$
38° 10′	$\sin \angle Q = (\cos 51° 45') \div (\cos 38° 10') = (.619)/(.786) \Rightarrow \angle Q = 52°$
42° 25′	$\sin \angle Q = (\cos 51° 45') \div (\cos 42° 25') = (.619)/(.739) \Rightarrow \angle Q = 57°$
47° 15′	$\sin \angle Q = (\cos 51° 45') \div (\cos 47° 15') = (.619)/(.679) \Rightarrow \angle Q = 66°$
49° 40′	$\sin \angle Q = (\cos 51° 45') \div (\cos 49° 40') = (.619)/(.647) \Rightarrow \angle Q = 73°$
51° 10′	$\sin \angle Q = (\cos 51° 45') \div (\cos 51° 10') = (.619)/(.627) \Rightarrow \angle Q = 81°$
51° 42′	$\sin \angle Q = (\cos 51° 45') \div (\cos 51° 42') = (.619)/(.619) \Rightarrow \angle Q = 90°$
51° 35′	$\sin \angle Q = (\cos 51° 45') \div (\cos 51° 35') = (.619)/(.621) \Rightarrow \angle Q = 96°$

Inspection of the gnomonic and Mercator charts corroborates these headings.

B. At any two points the sines of the DL_0s from the nearest meridian where the course crosses the equator are directly proportional to the tangents of the latitudes.

First, prove Napier's rule # 3. Returning to the right figure on page 3:

$$\begin{aligned} \tan a &= DF/OF \\ &= EF/OF \bullet DF/EF \\ &= \sin b \bullet \tan A. \end{aligned}$$

So for points B_1 and B_2 in the great circle figure on page 15 ~~14~~,

$$\tan a_1 = \sin b_1 \bullet \tan \angle A$$

and $$\tan a_2 = \sin b_2 \bullet \tan \angle A,$$

so $$\tan a_1 \div \sin b_1 = \tan a_2 \div \sin b_2,$$

or $$\sin b_1 \div \sin b_2 = \tan a_1 \div \tan a_2.$$

That is, the sines of the DL_Os from the meridian of the nearest point L_{zero} where the course crosses the equator are proportional to the tangents of the latitudes.

If the chart does not show the meridian of L_{zero} it can be located by adding or subtracting 90° from the meridian where the course latitude = L_{max}, and vice-versa.

Does this actually prove out in the sail from St. Augustine to Bull Light? Yes. First, determine the longitude at the point where a = L_{max}. By inspection of NO 17, L_{max} = 51° 45′N at L_O = 17° 00′W. The near-equivalency of the right column below, reckoned with a slide rule, verifies the point:

L_O	DL_O	L	tan L/cos DL_O
81° 19′	63° 19′	29° 54	(.576)/(.449) = 1.28
80° 00′	62° 00′	31° 00	(.601)/(.470) = 1.28
70° 00′	52° 00′	38° 10	(.786)/(.616) = 1.27
60° 00′	42° 00′	42° 25	(.914)/(.743) = 1.28
50° 00′	32° 00′	47° 15	(1.08)/(.85) = 1.27
40° 00′	22° 00′	49° 40	(1.18)/(.93) = 1.27
30° 00′	12° 00′	51° 10	(1.24)/(.98) = 1.27
20° 00′	2° 00′	51° 42	(1.27)/(1.00) = 1.27
10° 18′	7° 42′	51° 35	(1.26)/(.99) = 1.26

C. At any two points the cosines of the DL_Os from the meridian of L_{max} are directly proportional to the tangents of the latitudes.

The same rule holds for the cosines of the DL_Os from L_{max}. The sine of any angle = the cosine of its complement. So in the great circle figure on page ~~13~~ 14 the DL_O between B_1 and M is the complement of the DL_O between B_1 and A. The same is true for B_2, B_3, ... ∴ the previous section also proves that the cosines of the DL_Os from B_1 and B_2 to M are proportional to the latitudes of B_1 and B_2.

D. Distance to destination.

Page 14-7 of the manual claims the distance from St. Augustine is 3332 nm. The quickest way to verify this is with the law of cosines. Applying it yields

$$\begin{aligned}
\cos D &= \sin L_1 \bullet \sin L_2 + \cos L_1 \bullet \cos L_2 \bullet \cos DL_O \\
&= \sin 29^\circ\ 54' \bullet \sin 51^\circ\ 35' \\
&\quad + \cos 29^\circ\ 54' \bullet \cos 51^\circ\ 35' \bullet \cos 71^\circ\ 01' \\
&= (.498)(.784) + (.867)(.621)(.326) \\
&= .390 + .176 \\
&= .566 \\
D &= 55^\circ\ 31' \\
&= 3331 \text{ nm.}
\end{aligned}$$

The next best way is to use the great circle sailing chart, NO17. The legend explains a plotting method for measuring distances. Following that I come up with 3328 nm.

A more cumbersome method is to use Napier's rule # 5. Since latitude is measured from the equator, the rule can only give you the distance of a point from L_{zero}, the equator crossing point.

First, locate L_{zero}. Using rule # 3, for any two points B_1 and B_2, whether on the same side of the equator or not,

$$\begin{aligned}
\tan \angle A &= \tan a_1/\sin b_1 \\
&= \tan a_2/\sin b_2.
\end{aligned}$$

Since the longitudes are known, DL_O is $(81^\circ\ 19') - (10^\circ\ 18') = 71^\circ\ 01'$, so $b_2 = b_1 + 71^\circ\ 01'$. So, L_{zero} not being between B_1 and B_2,

$$\tan a_1/\sin b_1 = \tan a_2/\sin (b_1 + 71^\circ\ 01'),$$

and you can solve for b_1, which then also gives you b_2:

$$\begin{aligned}
\tan 29^\circ\ 54'/\sin b_1 &= (\tan 51^\circ\ 35')/\sin (b_1 + 71^\circ\ 01') \\
(.575)/\sin b_1 &= (1.26)/\sin (b_1 + 71^\circ\ 01') \\
\sin b_1 &= (.575)/(1.26) \bullet \sin(b_1 + 71^\circ\ 01') \\
&= (.456) \bullet \sin(b_1 + 71^\circ\ 01').
\end{aligned}$$

Here, we use the trigonometric identity for the sine of the sum of two angles shown at the end of this paper:

$$\begin{aligned}
\sin b_1 &= (.456) \bullet \\
&\quad (\sin b_1 \bullet \cos 71^\circ\ 01' + \\
&\quad \cos b_1 \bullet \sin 71^\circ\ 01') \\
&= (.456)\bullet ((.326) \bullet \sin b_1 + (.946) \bullet \cos b_1) \\
&= .149 \sin b_1 + .432 \cos b_1 \\
.851 \sin b_1 &= .432 \cos b_1 \\
\sin b_1/\cos b_1 &= .432/.851 \\
\tan b_1 &= .507 \\
b_1 &= 26.9^\circ \\
&= 26^\circ\ 53'.
\end{aligned}$$

The great circle crosses the equator at L_{zero} 26° 53′ west of St. Augustine. By rule # 5, the distance from there to St. Augustine can now be determined:

$$\begin{aligned}
\cos c_{SA} &= \cos a \bullet \cos b \\
&= \cos 29^\circ\ 54' \bullet \cos 26^\circ\ 53' \\
&= (.867)(.892) \\
&= .774 \\
c_{SA} &= 39^\circ\ 18'.
\end{aligned}$$

The distance from there to the Bull Light is determined by:

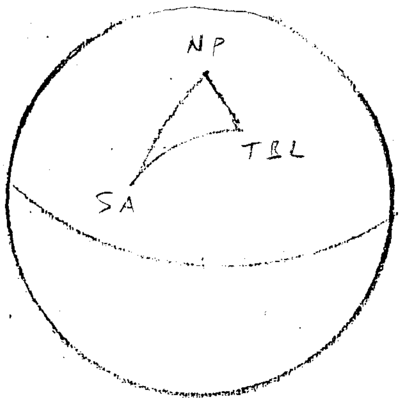
$$\begin{aligned}
\cos c_{TBL} &= \cos a \bullet \cos b \\
&= \cos 51^\circ\ 35' \bullet \cos (26^\circ\ 53' + 71^\circ\ 01') \\
&= \cos 51^\circ\ 35' \bullet \cos (97^\circ\ 54') \\
&= (.62)(-.137) \\
&= -.085 \\
c_{TBL} &= 94.89^\circ \\
&= 94^\circ\ 53'.
\end{aligned}$$

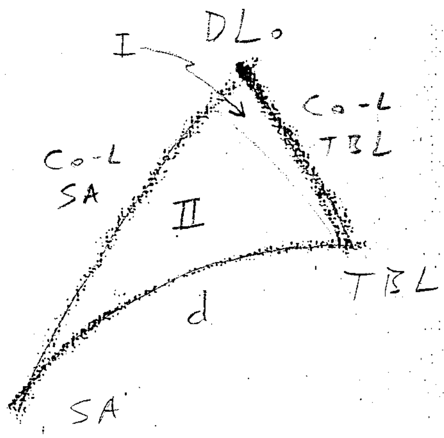
So

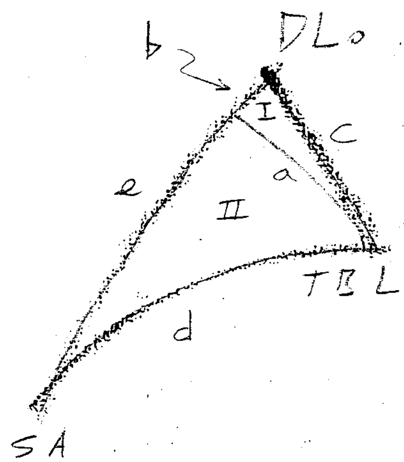
$$\begin{aligned}
D &= 94^\circ\ 53' - 39^\circ\ 18' \\
&= 55^\circ\ 35' \\
&= 3335 \text{ nm},
\end{aligned}$$

pretty close to the results using the law of cosines or NO17.

There is a fourth way to compute distance, and that is to use the same Napier formulas ## 1, 2, and 5 used to reduce a sight to H_C. That is, form a spherical triangle of the departure and destination points with the north pole, divide it into 2 right triangles, and solve them one at a time. For clarity, re-letter triangles I and II as shown in the right figure:







In triangle I:

$$\begin{aligned}
\sin a &= \sin c \cdot \sin \angle DL_0 \\
&= (\sin 38^\circ\ 25')\cdot(\sin 71^\circ\ 01') \\
&= (.62)(.946) \\
&= .587 \\
a &= 35^\circ\ 55',
\end{aligned}$$

and

$$\begin{aligned}
\tan b &= \tan c \cdot \cos \angle DL_0 \\
&= (\tan 38^\circ\ 25')\cdot(\cos 71^\circ\ 01') \\
&= (.794)\cdot(.323) \\
&= .256 \\
b &= 14^\circ\ 50'.
\end{aligned}$$

In triangle II:

$$\begin{aligned}
e &= (\text{Co-L SA}) - b \\
&= 60^\circ\ 06' - 14^\circ\ 50' \\
&= 45^\circ\ 16',
\end{aligned}$$

$$\begin{aligned}
\cos d &= \cos a \cdot \cos e \\
&= (\cos 35^\circ\ 55')\cdot(\cos 45^\circ\ 16') \\
&= (.81)\cdot(.704) \\
&= .57,
\end{aligned}$$

and

$$\begin{aligned}
d &= 55^\circ\ 15' \\
&= 3315\ \text{nm},
\end{aligned}$$

again, pretty close to the first 3 results.

Taking one more example, suppose a long great circle trip from America to Australia, ¾ of the way round the world through the southern ocean. The starting and ending coordinates as noted in Appendix E are:

Barnegat Inlet Light
New Jersey

39° 45′N
74° 06′W

Fremantle, Australia

32° 03′S
115° 45′E

The fourth method just noted -- the one analogous to reduction to H_C -- is not suited to a trip over 180° of circumference; Napier's rules have been proved (in this paper) only for spherical triangles with sides which are minor arcs. We don't have a gnomonic chart for such a great distance. Let's try the law of cosines. This is permissible even though the course > 180°, since for any angular distance D, cos D = cos (360° - D).

$$\begin{aligned}\cos D &= \cos L_1 \bullet \cos (90° + L_2) \\ &\quad + \sin L_1 \bullet \cos (90° + L_2) \bullet \cos DL_O \\ &= \cos 50° 15' \bullet \cos 122° 03' \\ &\quad + \sin 50° 15' \bullet \sin 122° 03' \circ \cos 189° 51' \\ &= (.639)(-.531) + (.769)(.848)(-.985) \\ &= -.339 - .641 \\ &= -.980 \\ D &= 273° 40' \text{ or } 266° 20' \\ &= 16{,}420 \text{ nm or } 15{,}980 \text{ nm.}\end{aligned}$$

It is not easy to tell by inspection which of these answers is correct. But the method of locating the equatorial crossing point tells us it is the higher figure:

The DL_O of the departure and destination (B_1 and B_2) is 189° 51′. Using rule # 3, and noting that L_{zero} is between B_1 and B_2, locate the L_O of L_{zero}:

$$\begin{aligned}\tan a_1 \div \sin b_1 &= \tan a_2 \div \sin b_2 \\ \tan 39° 45' \div \sin b_1 &= \tan 32° 03' \div \sin (189° 51' - b_1) \\ .832 \div \sin b_1 &= .626 \div \sin (189° 51' - b_1) \\ \sin b_1 &= .832/.626 \bullet \sin (189° 51' - b_1) \\ &= 1.33 \bullet \sin (189° 51' - b_1) \\ &= 1.33 \bullet (\sin 189° 51' \bullet \cos b_1 \\ &\quad - \cos 189° 51' \bullet \sin b_1) \\ &= 1.33 \bullet (- (.171) \bullet \cos b_1 \\ &\quad + (.985) \bullet \sin b_1) \\ &= 1.305 \sin b_1 - .227 \cos b_1 \\ -.305 \sin b_1 &= -.227 \cos b_1 \\ \sin b_1 \div \cos b_1 &= .227 \div .305 \\ \tan b_1 &= .744 \\ b_1 &= 36° 40'\end{aligned}$$

east of Barnegat Light. L_{zero} is $\therefore$ at $L_O = 37° 26'W$. This is also 153° 11′ west of Fremantle. Using rule # 5 the great circle distance from there to Barnegat Light is

$$\begin{aligned}
\cos c_{BL} &= \cos a_{BL} \bullet \cos b_{BL} \\
&= \cos 39° 45' \bullet \cos 36° 40' \\
&= (.769)(.802) \\
&= .616 \\
c_{BL} &= 52° 00' \\
&= 3120 \text{ nm.}
\end{aligned}$$

The great circle distance from there to Fremantle is

$$\begin{aligned}
\cos c_F &= \cos a_F \bullet \cos b_F \\
&= \cos 153° 11' \bullet \cos 32° 03' \\
&= (-.892)(.848) \\
&= -.756 \\
c_F &= 220° 53' \\
&= 13{,}253 \text{ nm,}
\end{aligned}$$

and

$$\begin{aligned}
c &= c_{BL} + c_F \\
&= 3120 + 13{,}253 \\
&= 16{,}373 \text{ nm.}
\end{aligned}$$

This is closest to the figure of 16,420 developed with the law of cosines.

Does this course go only on the ocean? Or does it bump into South America and Africa? To see that it does not, determine its latitude when passing by the nearest points in Brazil and South Africa. These are Cabo Calcanhar Light and the Cape of Good Hope. By Appendix E the coordinates of Cabo Calcanhar are L 5° 10′S and L_O 35° 29′W, 1° 57′ east of L_{zero}. By rule # 3:

$$\begin{aligned}
\tan 39° 45' \div \sin 36° 40' &= \tan a_{CCL} \div \sin 1° 57' \\
\tan a_{CCL} &= (.832)(.035)/(.597) \\
&= .0487 \\
a_{CCL} &= 2° 47',
\end{aligned}$$

2° 23′ north of Cabo Calcanhar.

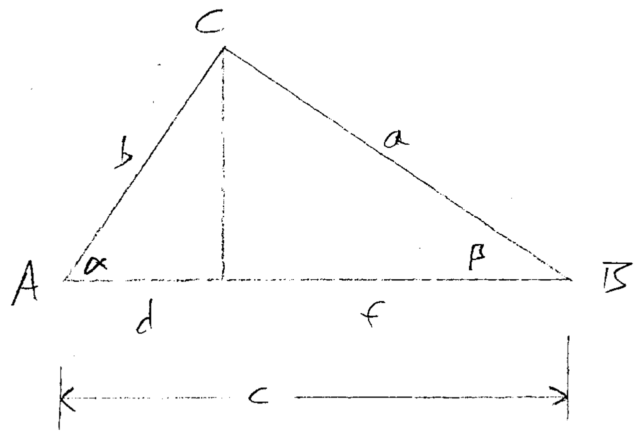
The Cape of Good Hope is at L 34° 20′S, L_O 18° 30′E, which is 55° 56′ east of L_{zero}. So,

$$\begin{aligned}
\tan 39° 45' \div \sin 36° 37' &= \tan a_{CGH} \div \sin 55° 56' \\
\tan a_{CGH} &= (.832)(.828)/(.597) \\
&= 1.152 \\
a_{CGH} &= 49° 05',
\end{aligned}$$

well south of the Cape.

IX. PLANE TRIGONOMETRY REVIEW

A. The Pythagorean Theorem: In a right triangle $c^2 = a^2 + b^2$.



In the figure,

$$\begin{aligned} d/b &= \cos\alpha \\ &= b/c \\ &= b/(d + f) \\ b^2 &= d(d + f) \\ &= d^2 + df \end{aligned}$$

$$\begin{aligned} f/a &= \cos\beta \\ &= a/c \\ &= a/(d + f) \\ a^2 &= f(d + f) \\ &= f^2 + df \end{aligned}$$

$$\begin{aligned} a^2 + b^2 &= d^2 + 2df + f^2 \\ &= (d + f)^2 \\ &= c^2. \end{aligned}$$

B. Definitions and graphs of the trigonometric functions.

This figure and table are taken from Gondin and Sohmer, "Advanced Algebra and Calculus Made Simple" (Doubleday & Co, Garden City, 1959), page 17.

$\sin\theta = y/r \qquad \csc\theta = r/y$

$\cos\theta = x/r \qquad \sec\theta = r/x$

$\tan\theta = y/x \qquad \cot\theta = x/y$

Generalized definitions of the trigonometric functions of any angle $\theta = QOP$ as in Figure 2 and table.

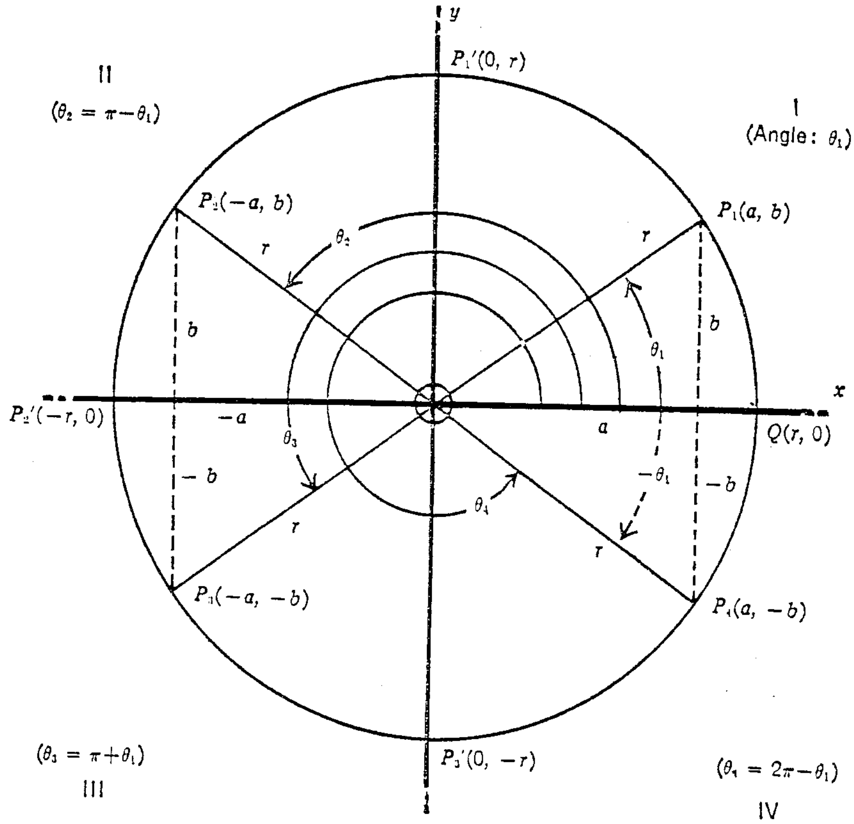


Figure 2

TABLE: Typical Values of Trigonometric Functions

Quad't		I		II		III		IV	
θ	0	θ_1	$\pi/2$	θ_2	π	θ_3	$3\pi/2$	θ_4	2π
P (x,y)	Q $(r,0)$	P_1 (a,b)	P'_1 $(0,r)$	P_2 $(-a,b)$	P'_2 $(-r,0)$	P_3 $(-a,-b)$	P'_3 $(0,-r)$	P_4 $(a,-b)$	Q $(r,0)$
Sin θ	0	b/r	1	b/r	0	$-(b/r)$	-1	$-(b/r)$	0
Cos θ	1	a/r	0	$-(a/r)$	-1	$-(a/r)$	0	a/r	1
Tan θ	0	b/a	$\pm\infty$	$-(b/a)$	0	b/a	$\pm\infty$	$-(b/a)$	0
Cot θ	$\mp\infty$	a/b	0	$-(a/b)$	$\mp\infty$	a/b	0	$-(a/b)$	$\mp\infty$
Sec θ	1	r/a	$\mp\infty$	$-(r/a)$	-1	$-(r/a)$	$\mp\infty$	r/a	1
Csc θ	$\mp\infty$	r/b	1	r/b	$\pm\infty$	$-(r/b)$	-1	$-(r/b)$	$\mp\infty$

These graphs are taken from Gondin and Sohmer, page 18.

REFERENCE GRAPHS OF TRIGONOMETRIC FUNCTIONS

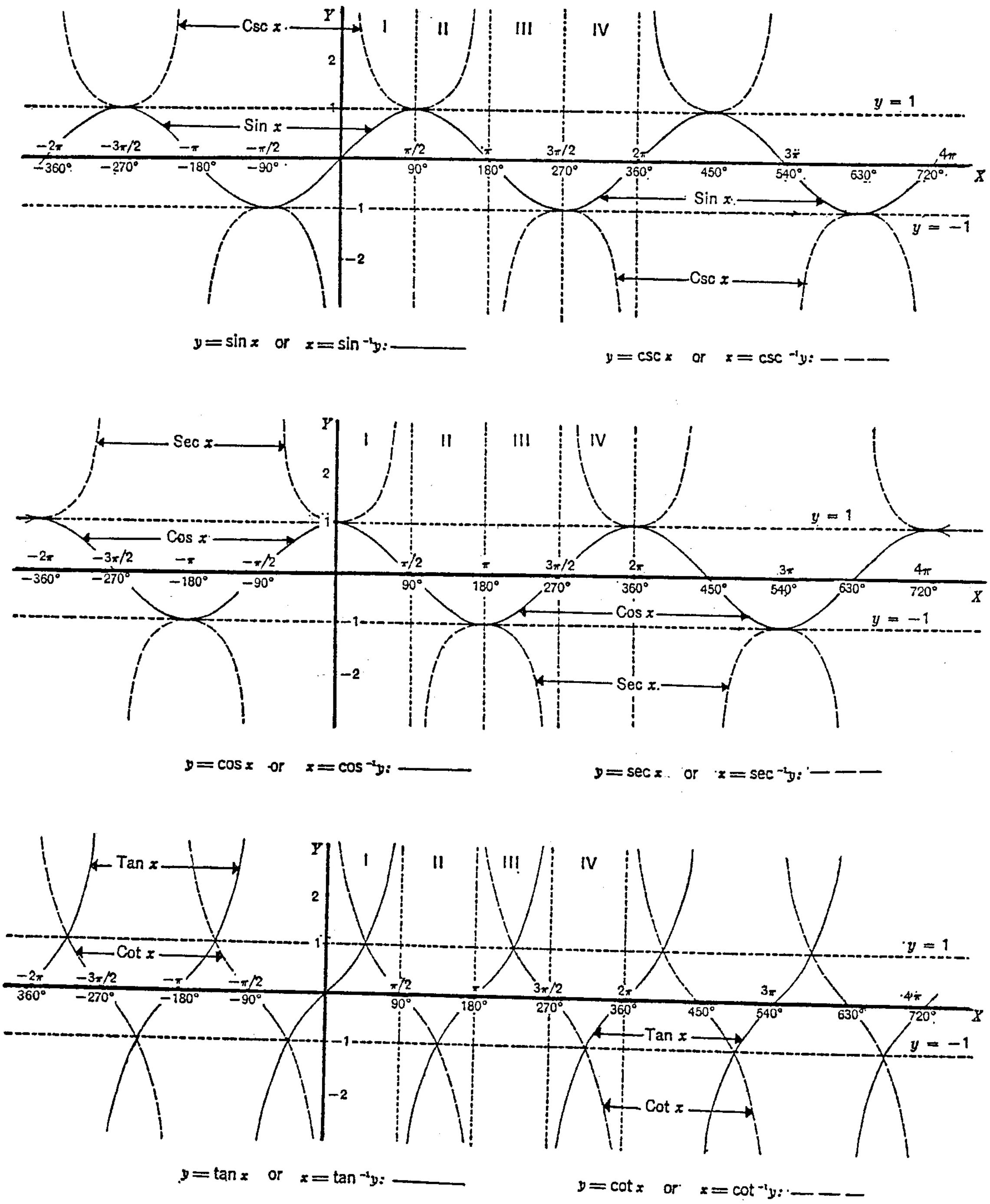
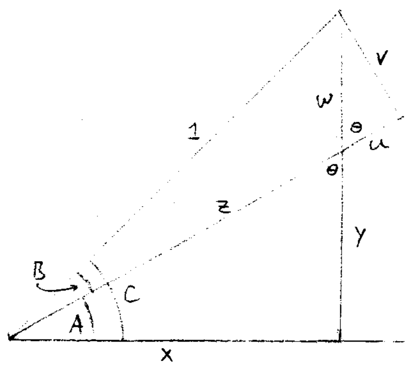
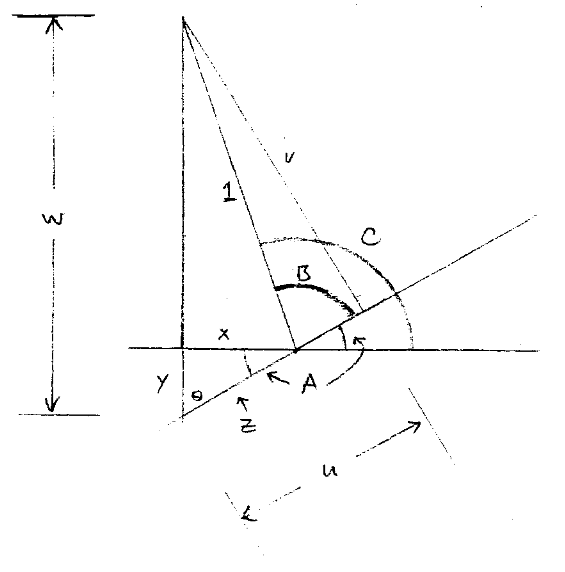


Figure 3

C. $\text{Sin}\ (A \pm B) = (\text{Sin}\ A \bullet \text{Cos}\ B) \pm (\text{Cos}\ A \bullet \text{Sin}\ B)$.

We will prove this for all positive angles < 180°.





In the left figure:

$$
\begin{aligned}
\sin A \bullet \cos B + \cos A \bullet \sin B &= (y/z)(z + u) + (x/z)(v) \\
&= y + (uy/z) + (vx/z) \\
&= y + \\
&\quad (y/z)(w)(\cos \Theta) + (x/z)(w)(\sin \Theta) \\
&= y + (w/z)(y \bullet \cos \Theta + x \bullet \sin \Theta) \\
&= y + (w/z)(y \bullet y/z + x \bullet x/z) \\
&= y + w \bullet (y^2 + x^2)/(z^2) \\
&= y + w \\
&= \sin C \\
&= \sin (A + B)
\end{aligned}
$$

and

$$
\begin{aligned}
\sin C \bullet \cos A - \cos C \bullet \sin A &= (w + y)(x/z) - (x)(y/z) \\
&= (wx/z) + (yx/z) - (xy/z) \\
&= w \bullet (x/z) \\
&= w \bullet \sin \Theta \\
&= w \bullet (v/w) \\
&= v \\
&= \sin B \\
&= \sin (C - A)
\end{aligned}
$$

In the right figure:

$$
\begin{aligned}
\sin A \cdot \cos B + \cos A \cdot \sin B &= (y/z)\cdot(u - z) + (x/z)\cdot(v) \\
&= (u\cdot y/z) + (v\cdot x/z) - y \\
&= (w\cdot\cos\Theta)\cdot(y/z) \\
&\quad + (w \cdot \sin\Theta)\cdot(x/z) - y \\
&= (w/z)\cdot(y \cdot \cos\Theta + x \cdot \sin\Theta) - y \\
&= (w/z)\cdot(y\cdot y/z + x\cdot x/z) - y \\
&= w\cdot(y^2 + x^2)/(z^2) - y \\
&= w - y \\
&= \sin C \\
&= \sin(A + B),
\end{aligned}
$$

and

$$
\begin{aligned}
\sin C \cdot \cos A - \cos C \cdot \sin A &= (w - y)(x/z) - (-x)(y/z) \\
&= (wx/z) - (yx/z) + (xy/z) \\
&= w\cdot(x/z) \\
&= w \cdot \sin\Theta \\
&= w\cdot(v/w) \\
&= v \\
&= \sin B \\
&= \sin(C - A).
\end{aligned}
$$

D. $\text{Cos}(A \pm B) = (\text{Cos } A \cdot \text{Cos } B) \mp (\text{Sin } A \cdot \text{Sin } B)$.

In the same left figure of the previous section:

$$
\begin{aligned}
\cos A \cdot \cos B - \sin A \cdot \sin B &= (x/z)(z + u) - (y/z)(v) \\
&= (zx + ux - vy)/z \\
&= x + (ux/z) - (vy/z) \\
&= x + (x/z)(u) - (y/z)(v) \\
&= x + u \cdot \sin\Theta) - v \cdot \cos\Theta) \\
&= x + uv/w - uv/w \\
&= x \\
&= \cos C \\
&= \cos(A + B)
\end{aligned}
$$

and

$$
\begin{aligned}
\cos C \cdot \cos A + \sin C \cdot \sin A &= (x)(x/z) + (w + y)(y/z) \\
&= (x^2 + y^2 + wy)/z \\
&= (z^2 + wy)/z \\
&= z + w \cdot \cos\Theta \\
&= z + w(u/w) \\
&= z + u \\
&= \cos B \\
&= \cos(C - A).
\end{aligned}
$$

In the right figure:

$$\begin{aligned}\cos A \bullet \cos B - \sin A \bullet \sin B &= (x/z)\bullet(u - z) - (y/z)\bullet(v)\\ &= (ux/z) - (vy/z) - x\\ &= (u\bullet\sin\Theta) - (v \bullet \cos\Theta) - x\\ &= u(v/w) - v\,(u/w) - x\\ &= -x\\ &= \cos C\\ &= \cos(A + B),\end{aligned}$$

and

$$\begin{aligned}\cos C \bullet \cos A + \sin C \bullet \sin A &= (-x)(x/z) - (w - y)(y/z)\\ &= (-x^2 - y^2 + wy)/z\\ &= (wy - z^2)/z\\ &= w(y/z) - z\\ &= w \bullet \cos\Theta - z\\ &= w(u/w) - z\\ &= u - z\\ &= \cos B\\ &= \cos(C - A).\end{aligned}$$

E. Napier's other rules.

Napier's last 4 rules have not been proved. For completeness, here they are. Rules 7-9 are simply the flip sides of rules ## 2-4, proved the same way as rule # 6. As for rule # 10, by rules ## 4, 5, and 9:

$$\begin{aligned}\cos c &= \cos a \bullet \cos b\\ &= \cos\angle A/\sin\angle B \bullet \cos\angle B/\sin\angle A\\ &= \cos\angle A/\sin\angle A \bullet \cos\angle B/\sin\angle B\\ &= \cot\angle A \bullet \cot\angle B.\end{aligned}$$

F. The law of sines: in any spherical triangle the sines of the angles are proportional to the sines of the opposite sides.

In the triangle on page 10,

$$\begin{aligned}\sin a \bullet \sin C &= \sin f\\ &= \sin c \bullet \sin A\\ \sin a/\sin A &= \sin c/\sin C.\end{aligned}$$

If the perpendicular were drawn from a to CB the same process leads to $\sin b/\sin B = \sin c/\sin C$ when it is recalled that $\sin B = \sin (180^\circ - B)$. This was to be proved.