General Algorithm for Extracting Approximate Square Roots of Real Numbers

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Last year in casual conversation with friends one of them wondered aloud what was the square root of 500. I thought about it for a few seconds and gave the answer within a couple of decimal points.

How? The following may help answer. In middle school before the days of calculators, by rote everybody in the class learned the algorithm for extracting square roots of real numbers just with pencil and paper.

Here is the method for any real number > 1. It is easiest to visualize by working through the example of $\sqrt{700}$ typed below, while you read.

> Mark off digit places by 2's leftward and rightward from the decimal point, leading with a 0 if necessary to make the left-most a pair, and adding a following 0 to the right-most digit if necessary to make it a pair. Find the highest digit whose square is less than the left-most pair. This digit multiplied by 10^{n-1}, (where n = the number of pair the left-most pair it is to the left of the decimal point) = the first partial square root.
>
> To get the second partial square root, multiply the first partial by 20, find a second high digit, the highest one such that when the foregoing product is added to it the resulting sum is less than the four-digit combination of the *two* left-most pair of digits. 10^{n-2} times the second high digit + the first partial square root = the second partial square root.
>
> To get the third partial square root and the ones after that, repeat the process hundreds of times or as many as you care to. The last one you do is the approximate square root of the original number.

Try making up your own example, say 123,456 or 9876.54321. Then check it by multiplying your answer by itself.

What about positive numbers < 1? For them the first partial is reckoned the same way as the first partial seen above. For the second and following partials the algorithm is a bit different but quite similar. Try an example, for instance by using one of the examples you already tried but with the decimal point moved all the way to the left.

For 1 itself, the square root of course = 1. (Square roots of negative numbers are imaginary.)

Query: Does this really work or is it a lie? And are the results useful?

They are. I puzzled this out around the time I was divorced in 2004. What the algorithm does is compute ever closer approximations of the desired square root, from below. If you are lucky, eventually it will hit dead on.

Consider a lemma: For all x, y:

$$\begin{aligned}(10x + y)^2 &= (10x)^2 + 20xy + y^2 \\ &= (10x)^2 + y(20x + y) \\ y(20x + y) &= (10x + y)^2 - (10x)^2\end{aligned}$$

Now, solve $\sqrt{p}$, where p is any positive real number, this way:

First, find any number a such that

$$a^2 \leq p$$

Then find any number b such that (using the lemma),

$$\begin{aligned}b(2a + b) &\leq p - a^2 \\ a^2 + 2ab + b^2 &\leq p \\ (a + b)^2 &\leq p\end{aligned}$$

The same way, find any number c such that

$$\begin{aligned}c[(2(a + b) + c] &\leq p - (a + b)^2 \\ (a + b)^2 + c[(2(a + b) + c] &\leq p \\ (a + b)^2 + 2c(a + b) + c^2 &\leq p \\ (a + b + c)^2 &\leq p\end{aligned}$$

Again, find any number d such that

$$\begin{aligned}d[(2(a + b + c) + d] &\leq p - (a + b + c)^2 \\ (a + b + c)^2 + d[(2(a + b + c) + d] &\leq p \\ (a + b + c)^2 + 2d(a + b + c) + d^2 &\leq p \\ (a + b + c + d)^2 &\leq p\end{aligned}$$

Continue in like manner. At some point you may actually find a number i such that

$$(a + b + c + d + \ldots + i)^2 = p .$$

If so, stop. You have won. If not, continuing similarly to add numbers will produce sums whose squares approach p *from below.*

qed

This is the same algorithm we learned in middle school, with an added bonus: It works regardless whether a, b, … are the highest possible digits, or even digits at all.

As an example, suppose p = 700, illustrated below (in pencil in 2004):

$a = 13$ then $p - (a)^2$ = a small number (relative to 700), 531.
$b = 1\frac{1}{2}$ then $p - (a + b)^2$ = a smaller number, $489\frac{3}{4}$
$c = 10$ then $p - (a + b + c)^2$ = an even smaller number, $99\frac{3}{4}$
$d = 1\frac{1}{2}$ then $p - (a + b + c + d)^2$ = an even smaller number, 24
$e = \frac{1}{4}$ then $p - (a + b + c + d + e)^2$ = an even smaller number, $10^{15}/_{16}$.
…

In other words as you continue adding numbers, $(a + b + c + d + e + \ldots)^2$ gets closer and closer to p from underneath. In the end, under the middle school algorithm or this general one, $\sqrt{700}$ turns out to be vanishingly more than 26.4575131..... (26.4575131 times itself exactly = 699.99999943667161 .)

The key insight is the lemma. A further bonus: The algorithm works whether a, b, … are rational (like 5, 9474495, $^4/_7$, $^{9474495}/_{399601}$), irrational (like $\sqrt{2}$ or φ), imaginary (product of $\sqrt{-1}$), complex (sum of a real and an imaginary), transcendental (like π or e), or any algebraic number which is positive. Nice.

The middle school method for $\sqrt{700}$:

```
                   2  6 .  4   5   7   5   1   3   1   .     .    .
                 √07 00 . 00 00 00 00 00 00 00 00 00 00
                  04
          46 |03 00
             |02 76
         524     |24   00
                 |20   96
        5285     |03   04 00
                 |02   64 25
       52907          |39 75 00
                      |37 03 49
      529145          |02 71 51 00
                      |02 64 57 25
     5291501              |06 93 75 00
                          |05 29 15 01
    52915023              |01 64 59 99 00
                          |01 58 74 50 69
   529150261                  |05 85 48 31 00
                              |05 29 15 02 61
  . . . . . . . . . .             |56 33 28 39 00
                                  |  .  ..   ..   ..   ..
 . . . . . . . . . . .              |   ..  ..   ..   ..  00
                                    |    .  ..   ..   ..   ..
. . . . . . . . . . . . .             | ..  ..   ..  ..  ..  00
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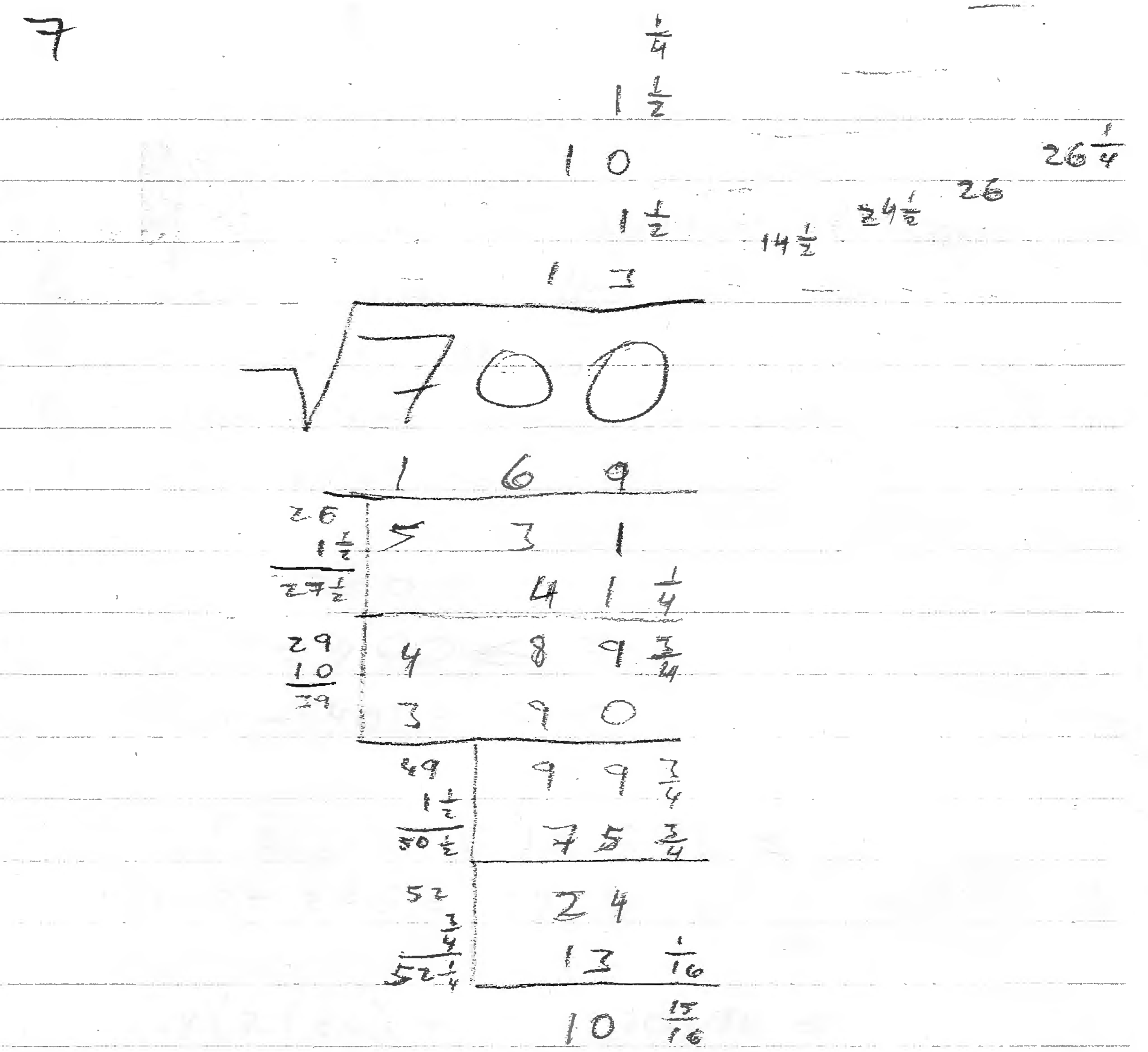
The general method for the square root of 700, worked out in 2004



Notice the approximate square roots on the top line are successively closer approximations, each larger than the preceding, and each $< \sqrt{700}$: 13, $14\frac{1}{2}$, $24\frac{1}{2}$, 26, $26\frac{1}{4}$.
Notice also how the remainders fit in:

$$
\begin{aligned}
700 &= (13)^2 + 531 \\
&= (14\tfrac{1}{2})^2 + 489\tfrac{3}{4} \\
&= (24\tfrac{1}{2})^2 + 99\tfrac{3}{4} \\
&= (26)^2 + 24 \\
&= (26\tfrac{1}{4})^2 + 10\tfrac{15}{16}
\end{aligned}
$$